Probability Theory

First module, 2024/2025

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Course information

Course Website:

Instructor's Office Hours:

Class Time:

Room Number:

TAs: [Names and contact information]

Course description

"Probability Theory" is the first course in the series of the courses Probability − Statistics − Econometrics. The main purposes of the course are the following:

- to learn the bases of probability theory;
- to set the theoretical tools for studying mathematical statistics and econometrics;
- to introduce the principles of statistical modeling.

The course consists of the five main parts:

1. The foundations of probability theory. The simple probabilistic schemes.
2. Random variables and random vectors.
3. Moment generation functions. Multi-dimensional normal distribution.
4. Limit theorems: Law of large numbers, Central limit theorem. Asymptotic normality.

Course requirements, grading, and attendance policies

Pre requisites: standard courses of calculus and linear algebra.
There are fourteen lectures (28 hours) and seven classes (14 hours). Six weekly home assignments are suggested that account 20% of the final grade. Solutions are distributed. The final exam (written format) accounts for 80% of the final grade. One A4 list of paper is permitted. **If exam grade is less than 25 points the final mark is "failed" regardless of other marks.**
The make-up exam has the same format as the final exam.

Course contents

I. Foundations of probability theory. Simple probabilistic scheme (3 lectures)

1. Experiment with random outcomes.
2. Probability space. Random events. Operations on events.
3. Finite probability spaces. Classical probability. Elements of combinatorics.
4. Geometric probability.
5. Conditional probability. Full probability formula. Bayes rule. Independent events.
6. General probability space. Sigma-algebra of random events. Probability, its properties. Continuation of a probability from algebra of events to the generating sigma-algebra.

II. Random variables and random vectors (5 lectures)

1. Random variable, its distribution. Discrete and continuous random variables.
2. Numeric characteristics of random variable (expectation, variance, median, etc.).
3. Examples of discrete and continuous random variables.
4. Random vectors, their distributions.
5. Independent random variables. Covariance, correlation.
6. Conditional distribution.

III. Moment generation functions. Multi-dimensional normal distribution (2 lectures)

1. Definition of Moment generation functions for random variables and random vectors. Properties of Moment generation functions.
2. Multi-dimensional normal distribution.
3. Chi–square distribution, Student distribution, Fisher distribution.
4. Fisher's lemma.

IV. Limit theorems (2 lectures)

1. Convergence of the sequences of random variables: almost surely, in the mean, in probability, in distribution (weak).
2. Convergence of characteristic functions and convergence of distributions.
3. Tchebyshev's inequality. Law of large numbers.
4. Central limit theorem. Normal approximation of Binomial and Poisson distributions.
5. Asymptotic normality.

Description of course methodology

Theoretical material given at lectures illustrates by numerous examples. Class teaching basically is devoted to the clarifying of the lectures' materials and to the solution of the problems.

Sample class problems.

Problem 1. Let X be a random variable with continuous cdf $F(x)$. Find the distribution of the random variable $Y = F(X)$.

Problem 2. Let X_1, X_2 be independent geometric random variables.

(a) Prove that the random variable $Y = \min(X_1, X_2)$ has geometric distribution.

(b) Find $E(Y)$.

Problem 3. Find expectation and variance of an exponential random variable.

Sample tasks for course evaluation

Sample Home Task problems

Problem 1. The device consists of two blocks. The lifetime (time before the break) has exponential distribution with parameters $\lambda_1 = \frac{1}{6}$, $\lambda_2 = \frac{1}{8}$, respectively, and these variables are independent. The device breaks down if at least one block does. What is the mean lifetime of the device?

Problem 2. Point is randomly selected at the interval [0, 1] of the *Ox* axis. Let *X* be a distance from this point to the point (0, 1). Find the distribution of the random variable *X*.

Problem 3. Let X_1, X_2 be independent Poisson random variables with parameters λ_1, λ_2 respectively.

(a) Find conditional distribution $p_n(k) = \Pr(X_1 = k \mid X_1 + X_2 = n),\ k = 0,1,\ldots,n$.

(b) Find $E(X_1 \mid X_1 + X_2 = n)$.

Sample exam problems.

Problem 1

A worker's skill is described by the random variable θ that may take values 0 or 1 equally likely. The worker's productivity at day $t = 1,2$ is $y_t = \theta + \varepsilon_t$, where $\varepsilon_t,\ t = 1,2$ are the production shocks. The random values $\varepsilon_1, \varepsilon_2$ are independent and take values 0 or 1 with the probabilities q and p, respectively, $p + q = 1$. They are also independent on θ. The values y_1, y_2 are observed. Find the *aposteriori* distribution of the worker's skill θ for all possible values of y_1, y_2.

Problem 2

Let $A_1 = [\xi_1, \eta_1]'$, $A_2 = [\xi_2, \eta_2]'$, $A_3 = [\xi_3, \eta_3]'$ are independent two-dimensional standard normal vectors, interpreted as the points on coordinate plane.

(a) Find the distribution of the length of the median A_1M_1 in the triangle $A_1A_2A_3$.

(b) Find the expectation of this length.

Problem 3

The integral $J = \int_0^1 e^x\,dx$ is calculated via Monte-Carlo method, i.e. *n* independent uniform on [0,1] random variables $x_1, \ldots, x_n$ are simulated and the integral *J* is estimated as $J_n = \frac{1}{n}\sum_{i=1}^{n} e^{x_i}$. Find such *n* that J_n differs from *J* not greater than $\varepsilon = 0.02$ with probability not less than 0.95.

Course materials

Required textbooks and materials

1. Sh. Ross (2009.) A First Course in Probability, Pearson, Prentice Hall,

2. B.V.Gnedenko (1988). Course on Probability Theory, Moscow: «Nauka» (in Russian).

Additional materials

1. A.NShiryaev. Probability, MCNMO, 2011 (in Russian)
2. V.Chistyakov (2000) Probability Theory (5th edition). Moscow, «Agar» (in Russian).

Academic integrity policy

Cheating, plagiarism, and any other violations of academic ethics at NES are not tolerated.